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.

Here we present new calculations of optimal parameters for Uda-Yagi ** plane antenna with elements parallel to ideally conducting Earth. Antenna consists of 6 cylindrical elements diameter 2a each, namely reflector, emitter and four directors with lengths slightly varying to give the desirable radiation pattern .

Calculations are aimed at optimizing of parameters (maximizing forward/back intensity ratio called "shield coefficient and/or direct wave intensity)

We're calculating current distributions in the reflector, emitter and all the directors which provide zero electric field on the side surfaces of them and turn into zeros at the edges. In other words we find a numerical solution of boundary-value problem for Maxwell equations, Zommerfeld's radiation condition included, for the Uda-Yagi antenna.

For all the numerical results hold current symmetries discussed below and reciprocity theorem.

Antenna parameters, i . e . lengths of elements and mutual distances are adjusted during iterations process , first iteration being the results of MMANA program implementation to the antenna .

Our calculations showed that the MMANA program applied to the Uda-Yagi antenna provides a system with forward intensity amplification of 25 times compared to single cylindrical antenna , both at height of 4 meters above ideal Earth, with electromotive

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force (later EMF) of external feed in the middle of emitter the same. Shield coefficient (i. e. forward/back amplitude ratio) amounts to 20 decibels for Uda-Yagi antenna with MMANA parameters and total intensity ratio for Uda-Yagi to single antenna approximately to 3 . Our iterative adjustment of lengths gave direct wave enhancement of 40 instead of 25, shield coefficient of approximately 49 decibels instead of 20 and total intensity ratio of slightly more than 4.

We split the reflector, emitter and directors into sets of infinitesimal current elements (dipoles) and calculate total longitudinal electric fields on the side surfaces of all such elements through spacings of elements with unknown infinitesimal currents. The number of infinitesimal currents to calculate is obviously equal to the number of elements with zero longitudinal field on side surface

So we arrive at the system of linear equations for unknown complex amplitudes of infinitesimal currents, having single solution , best represented in tensor form.

Tensor form of system of linear equations enables us to implement for the solution a procedure quite analogous to Gauss-Jordan elimination or Gaussian elimination with back substitution.

We formulated our linear system so as the latter appeared transparent and slightly easier and without any noticeable rounding errors in matrix calculations with 16-digits computer accuracy.

Linear system consists of 6 tensor equations, one for reflector, emitter and four directors, with 6 vectors of infinitesimal complex

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amplitudes of current elements (dipoles) to be determined and 6*5=30 matrices of coefficients calculating longitudinal electric fields or, more precisely, the distributions of induced local EMF. Currents produced by these electromotive forces produce fields zeroing on side boundaries of each element of Uda- Yagi antenna. Multiplication of coefficients in Gauss elimination is plainly replaced with matrix multiplication and division of coefficients - with calculating a reverse matrix. All matrices to be reversed appear to be perfectly conditioned ones so we did not need the regularization procedure. Suffice it to say that, Crout method with or without pivoting gave no difference. The solution thus obtained fitted the linear system with accuracy of 16 digits.

We now replace infinitesimal current elements with discrete ones with lengths determined by their number on each antenna element (chosen equal to some constant N for all elements i.e. reflector, emitter and four directors)

The i-th current element complex amplitude is equal to the Green's function (to be determined) of antenna element multiplied by local EMF with summation over all current elements (summation over index k) from 1 to N.

$$I_i = g_{ik} e_k \tag{0}$$

EMF in the point k on antenna element "m" created by all current elements of antenna element "n" ("m" and "n" denote any pair

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of antenna elements) equals to

$$e_k = f_{kj} I_j \qquad (1)$$

EMF matrix f_{kj} here represents EMF (i.e. field strength multiplied by discrete current element length) in the point k of antenna element "m" created by the current element in the point j of antenna element "n". To get the EMF in the point k of antenna element " m" by all the current elements on antenna element " n " the summation over index j is required.

From (0) and (1) taking an account of the associativity of matrix multiplication we see (" m " and " n " are notations of any pair of antenna elements)

$$I^m{}_i = (g_{ik} f_{kj}) I^n{}_j \qquad (2)$$

Value in round brackets represent current influence matrix of antenna element " n " imposed on antenna element " m "

Let us use the notation for antenna elements : r – reflector , e- emitter (i.e. center- fed antenna element) , f- first director , s- second director, t- third director, l- (last) - fourth director and two-letter names for current influence matrices-first letter denoting antenna element " m " in (2) and second letter denoting antenna element " n " in it.

Here we readily recall that in the matrix product $(g_{ik} f_{kj})$ first matrix is the antenna element " m " Green's function depending of element " m " only.

and second matrix is the mutual EMF matrix depending of mutual spacings and properties of both.

The reciprocity theorem states that when antenna elements m and n exchange, EMF matrix can be obtained by transposition.

Albeit this holds , we have no need to use it.

In terms of the notations taken, for example

$$lt_{ij} = (g_{ik} f_{kj}) \qquad (3)$$

here lt_{ij} is influence matrix representing part of current in i-th point of fourth (last) director induced by current in j-th point of third director First matrix in round brackets is Green's function of last director and second matrix is the mutual EMF matrix of both. In view of reciprocity theorem, following same notations, $tl_{ij} = (g_{ik} q_{kj})$ here $q_{kj} = f_{jk}$, because Green's function g_{ik} obeys the reciprocity theorem, too

Let us find an expression for EMF matrix. Suppose we neglect any azimuthal currents, so vector potential has only one component parallel to antenna element.

For the vector potential of current element, in obvious notations,

dA = I*dl /(c*r) , and for electric field vector

$$d\vec{E} = -\frac{1}{ik}(graddiv\vec{A} + k^2\vec{A}\,), \ [2] \qquad (4)$$

Let's place the center of polar coordinates in the center of current element with polar axis parallel to the element of antenna, then.

(υ is the polar angle)

$$divA = \cos\upsilon\left(-\frac{1}{r^2} + \frac{ik}{r}\right)\frac{\exp(ikr)}{c} I * h2$$

After simple calculations we get (h2 is the length of radiating current element)

$$E_r = 2\cos\upsilon \frac{I * h2}{c}\frac{\exp(ikr)}{r}\left(\frac{1}{r} + \frac{i}{kr^2}\right) \quad ;$$

$$E_\upsilon = \sin\upsilon\left(\frac{i}{kr^2} + \frac{1}{r} - ik\right)\frac{\exp(ikr)}{cr} I * h2$$

Longitudinal field $E = E_r \cos\upsilon - E_\upsilon \sin\upsilon$, that is

$$E_l = \frac{I_j * h2}{c}\frac{\exp(ikr)}{r}\left(\left(\frac{1}{r} + \frac{i}{kr^2}\right) * (3\cos^2\upsilon - 1) + ik\sin^2\upsilon\right)$$

The EMF in any current element of any other element of antenna if the product of field by length of that other current element, h1.

$$\varepsilon = f_{ij} = \frac{I_j * h2}{c}\frac{\exp(ikr)}{r}\left(\left(\frac{1}{r} + \frac{i}{kr^2}\right) * (3\cos^2\upsilon - 1) + ik\sin^2\upsilon\right) * h1 \qquad (5)$$

with r and υ determined by indexes i and j .

Complex amplitudes I_j are values to be determined

Next we come to the task of calculating the Green's function for antenna element parallel to ideal flat Earth at the height H, given the Green's function of that antenna element in the free space. Boundary conditions hold if image antenna at same " height " H below Earth and of opposite current direction added [2]

Let us seek for the Green's function in the form

$$g_{ik} = g^0{}_{ik} + x_{ik} \qquad (6)$$

with Green's function to be found in the LHS and given Green's function for free space and appendix x_{ik} (not necessarily small) caused by field of image antenna.

According to what has been said we get

$x_{ik} = R_{ij}(-g^{0}{}_{jk} - x_{jk})$ where $R_{ij} = g^{0}{}_{il} f_{lj}$

Matrix in round brackets is the Green's function of image antenna and f_{lj} - EMF matrix of image and the original antenna.

From the last equations we derive (δ_{ij} is Kronecker ' s symbol)

$(R_{ij} + \delta_{ij})x_{jk} = -R_{il} g^{0}{}_{lk}$ $\delta_{ij} x_{jk} = x_{ik} = -(R_{ij} + \delta_{ij})^{-1}(R_{jl} g^{0}{}_{lk})$

The matrix is round brackets in RHS of second equation is the reverse of that in round brackets in LHS of first equation.

Here we come to the closed linear system of tensor equations with 6 N-dimensional vectors of current complex amplitudes to be determined and 30 N * N matrices of influence as coefficients. First letter in the matrix name denotes the antenna element in which current distribution is calculated and the second letter denotes the antenna element the electric field of which produces this part of current . Each matrix is the Green's function matrix of first antenna element multiplied by the EMF matrix of first and second elements calculated above (3)

Letters in matrix names mean r - reflector, e- center-fed emitter,f- first director, s- second director, t- third director, l - the last (fourth) director

$$I_i^l = (l\,t)_{i\,j}\, I_j^t + (l\,s)_{i\,j}\, I_j^s + (l\,f)_{i\,j}\, I_j^f + (l\,r)_{i\,j}\, I_j^r + (l\,e)_{i\,j}\, I_i^e \qquad (7)$$

. $$I_i^t = (t\,l)_{i\,j}\, I_j^l + (t\,s)_{i\,j}\, I_j^s + (t\,f)_{i\,j}\, I_j^f + (t\,r)_{i\,j}\, I_j^r + (t\,e)_{i\,j}\, I_i^e \qquad (8)$$

$$. \quad I_i^s = (s\,l)_{ij}\, I_j^l + (s\,t)_{ij}\, I_j^t + (s\,f)_{ij}\, I_j^f + (s\,r)_{ij}\, I_j^r + (s\,e)_{ij}\, I_i^e \qquad (9)$$

$$. \quad I_i^f = (f\,l)_{ij}\, I_j^l + (f\,t)_{ij}\, I_j^t + (f\,s)_{ij}\, I_j^s + (|f\,r)_{ij}\, I_j^r + (f\,e)_{ij}\, I_i^e \qquad (10)$$

$$. \quad I_i^r = (r\,l)_{ij}\, I_j^l + (r\,t)_{ij}\, I_j^t + (r\,s)_{ij}\, I_j^s + (rf\,)_{ij}\, I_j^f + (r\,e)_{ij}\, I_i^e \qquad (11)$$

$$. \quad I_i^e = I^0{}_i + (el\,)_{ij}\, I_j^l + (et)_{ij}\, I_j^t + (es)_{ij}\, I_j^s + (e\,f)_{ij}\, I_j^f + (er)_{ij}\, I_i^r \qquad (12)$$

The additional current $I^0{}_i$ in equation 12 represent the KNOWN current produced by external EMF exerted over the emitter in the center of it.

The ideal Earth's influence is twofold: through the Green's function modified as above and additional EMF analogous to that calculated in (7) caused by image antenna. Both corrections are to be subtracted from original expressions cause of opposite current in the image antenna

Here we are at last at the crucial point of calculating the Green's function of a single conducting cylinder of length 2L and diameter $2a$ in free space. Notations taken are shown below: cylindrical coordinate system with Z axis being the axis of cylinder and zero (center) in the middle of cylinder axis.

$R = \sqrt{(z - z'\}^2 + a^2 + r^2 - 2ar\cos(\varphi - \varphi')}$ - the distance between two points, one of which with coordinates z' , φ' is on the cylinder's side surface

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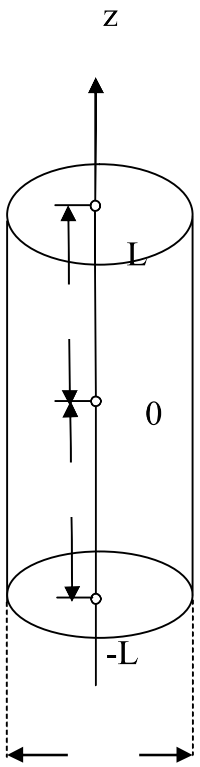

Let us consider an integral over the cylinder's side surface

$$V(z) = \int_0^{2\pi}\int_{-L}^{L} \frac{\exp(ikR)}{R} S(\varphi', z')\; a\; d\varphi'\, dz'$$ S = J/c j- surface current density

. V(z)- vector potential, f = exp(ikR)/R - Green's function of the

Helmholtz equation which takes the form (k= $2\pi/\lambda$ - wave number)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{\partial^2 f}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + k^2 f = 0$$

Separation of variables entail (w and s are parameters)

$$f = R(r)\; Z(z)\,\Phi(\varphi) \qquad \frac{1}{Z}\frac{d^2 Z}{d z^2} + w^2 = 0 \qquad \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} + s^2 = 0$$

Radial function R(r) obeys the equation

$$R''_{rr} + \frac{R'_r}{r} - \left(w^2 - k^2 + \frac{s^2}{r^2}\right) R = 0$$

It is limited on the z axis (r = 0) and tends to zero at infinity.

The Green's function of Helmholtz equation can be expanded into the series

$\frac{e^{ikR}}{R} = \sum_S \varepsilon_s \, f_s \, \cos(s(\varphi - \varphi'))$ with $\varepsilon_0 = 1$, $\varepsilon_1 = \varepsilon_2 = \cdots = 2$ Neumann coefficients

chosen so as we should have $f_s = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{ikR}}{R} \cos(s(\varphi - \varphi')) \; d(\varphi - \varphi')$

Let us neglect any azimuthal variations, in compliance to above .

Thus s = 0 , and the equation for R(r) becomes the modified Bessel equation

of zero order and all Fourier coefficients except for f_0 (r, z) denoted further

as f (r, z) , become zeroes. Separation of variables thus give

$$f = A\int_{-\infty}^{\infty} \cos w\,(z - z') \begin{Bmatrix} \mathrm{I}_0\left(\sqrt{w^2 - k^2}\; r\right) \; K_0\left(\sqrt{w^2 - k^2}\; a\right) \\ K_0\left(\sqrt{w^2 - k^2}\; r\right) \mathrm{I}_0\left(\sqrt{w^2 - k^2}\; a\right) \end{Bmatrix} dw \qquad \text{(13-0)}$$

where the modified Bessel solutions, i.e. Infeld function I_0 and

Mc Donald function K_0 are chosen so as f(r, z) be continuous at

$r = a$. A - a coefficient to be determined. Upper line in figure

brackets should be taken when $r < a$ lower line - when $r > a$.

Comparing f(z-z' , r) to the vector potential in the point $z = z' = r = 0$

created by dipoles uniformly distributed over the cylinder's basic circle

(Appendix 2) we get $A = \frac{1}{\pi}$, so finally

$$f = \frac{1}{\pi}\int_{-\infty}^{\infty} \cos w\,(z - z') \begin{Bmatrix} \mathrm{I}_0\left(\sqrt{w^2 - k^2}\; r\right) \; K_0\left(\sqrt{w^2 - k^2}\; a\right) \\ K_0\left(\sqrt{w^2 - k^2}\; r\right) \mathrm{I}_0\left(\sqrt{w^2 - k^2}\; a\right) \end{Bmatrix} dw \qquad \text{(13)}$$

This expression is exactly the averaged fundamental solution exp(ikR) / R in the terms of reverse Fourier transform of the value in figure brackets. Along with the distance R, it is an even function of the difference of z coordinates. This enable us with the possibility to find the current distribution U(z) and hence the vector potential V(z) in the form of Fourier series, which we treat as " Least Mean Square Deviation" series , taking no account of Dirichlet theorem (3)

When w < k we have $\sqrt{w^2-\mathrm{k}^2}=-i\sqrt{k^2-w^2}$ $\left(\sqrt{k^2-w^2}>0\right)$,

in view of Zommerfeld's radiation condition . Taking an account of

$\mathrm{I}_0(-ix)=J_0(x)$ $K_0(-ix)=i\frac{\pi}{2}\mathrm{H}_0^1(x)$ if w < k , we get when $(r<a)$

$$\mathrm{I}_0\left(\sqrt{w^2-k^2}\;r\right)\;K_0\left(\sqrt{w^2-k^2}\,a\right)=i\frac{\pi}{2}J_0\left(\sqrt{k^2-w^2}\,r\right)\;\mathrm{H}_0^1\left(\sqrt{k^2-w^2}\;a\right)$$

and when

$$(r>a):\;\mathrm{I}_0\left(\sqrt{w^2-k^2}\;a\right)\;K_0\left(\sqrt{w^2-k^2}\,r\right)=i\frac{\pi}{2}J_0\left(\sqrt{k^2-w^2}\;a\right)\;\mathrm{H}^1{}_0\left(\sqrt{k^2-w^2}\,r\right)$$

The above integral representation for vector potential V(z)

$\int_0^{2\pi}\int_{-L}^{L}\frac{\exp(ikR)}{R}S(\varphi',z')\;a\;d\varphi'\;dz'=V(z)$ S = J/c j- surface current density

with the not depended of azimuthal angle U(z) = S = I(z) / c

in left-hand side with I(z) = $2\pi a$ j(z) becomes

$$\int_{-L}^{L}f(z-z',a)\;U(z')\;dz'=V(z) \qquad (14)$$

with integrating along the cylinder axis length.

We will further omit here cylinder radius a as constant and consider this as an integral equation for current distribution $U(z')$.

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Let us write the vector potential as follows

$$V(z) = C^0 V^0(z) + C\cos kz + C'\sin kz \qquad (15)$$

The first term in the right hand side represents vector potential in the point with coordinate z caused by the current induced by the EMF applied in the point with coordinate z_0 equal to unity produced by some external field shaped as Dirac delta function, $E(z) = \delta(z - z_0)$. It is being calculated easily, on account of three simple relationships to follow.

First expression represents the zero balance between the external delta-shaped field and the field of current induced by external EMF equal to unity (integral of the delta-function). Second expression relates the induced field to the vector potential to be defined. Integrating in the vicinity of point z_0 we get the third expression

$$\mathrm{E}_{z_0} + \mathrm{E}^{\varepsilon}_{z_0} = 0 \qquad E = \frac{i}{k}\left(\frac{d^2V^0}{dz^2} + k^2V^0\right) \qquad \left.\frac{dV^0}{dz}\right|_{z=z_0+0} - \left.\frac{dV^0}{dz}\right|_{z=z_0-0} = ki$$

after which we find for the vector potential easily

$$V^0(z) = \frac{i}{2}\sin k|\ z - z_0\ | \qquad C^0 = 1 \qquad (16)$$

The independent solution of this, $V^{00}(z) = 0.5 * \exp(ik|\ z - z_0\ |)$ leads to the very same current distribution, as we shall see shortly afterwards.

It is essential for the technique to follow, that along with classical Fourier series in sin(nx) ,cos(nx), "half-integer" Fourier harmonics sin((n+0.5)x) cos((n+0.5)x) obey the orthogonality condition on the same interval .We draw this conclusion on the basis of formulas quite analogous to these describing the Fourier expansion outlined in [3]. Hence the half-integer Fourier harmonics can also be used for an expansion analogous to Fourier series with no zero term and Neumann factor equal to 2. The kernel of integral equation (14) is even function depending on the module of its argument , so we can divide current distribution and vector potential into even and odd parts and get the solution of (14) for both independently. For even part we use the ordinary Fourier expansion and for odd part - an expansion in half-integer Fourier-like series. Period of $f(z-z'))$ equals 4L, periods of U(z) и V(z) equal 2L.

. Let us first consider the even part of solution and write the Fourier expansions of the vector potential in the RHS , kernel of (14) and current distribution in the form

$$F_\nu = \int_0^{2L} f(z)\cos \nu gz\, dz \quad (17)$$

$$f = \frac{g}{\pi}\sum_{\nu=0}^{\infty}\varepsilon_\nu F_\nu \cos \nu g\,(z-z') \qquad g=\frac{\pi}{2L} \qquad (18)$$

$$V(z)=\sum_{\eta=0}^{\infty}\varepsilon_\eta V_\eta \cos \nu\,\frac{\pi}{L}z=\sum_{n=0}^{\infty}\varepsilon_n V_n \cos ngz \quad (19)$$

$$U(z)=\sum_{q}\varepsilon_q U_q \cos(qgz) \qquad \varepsilon_0=1 \quad \varepsilon_2=\varepsilon_4=\bullet\bullet\bullet=2 \quad (20)$$

Certain asymmetry in (18) compared to (19) and (20) is made for some future convenience. Values of n and q in (19) and (20) are even ones only. This is because of that U(z) and V(z) periods are a half of f(z) one.

On account of formula (13) and its reliance to Fourier transform of value in figure brackets there, taking an account of (17) we obtain .

$$F_\nu = I_0\left(\sqrt{(\nu^2 g^2 - k^2)}\, a\right) K_0\left(\sqrt{(\nu^2 g^2 - k^2)}\, a\right) + \Delta F_\nu \qquad \Delta F_\nu = -\int_{2L}^{\infty} f(z)\cos(\nu g z)\, d\, z \quad (21)$$

Modified Bessel functions were calculated with 16 digits computer accuracy in accordance with algorithms in [1] .

In calculation of the last integral in (21) we can make use of an approximation

$f(z) = \dfrac{e^{ikz}}{z}$ with simple corrections albeit we now bear in mind some much more sophisticated procedure.

$$\int_{2L}^{\infty} \frac{e^{ikz}}{z} \cos \nu g z\, dz \qquad \cos \nu g z = \frac{1}{2}\left(e^{i\nu g z} + e^{-i\nu g z}\right)$$

$$\int_{2L}^{\infty} \frac{e^{iz(k+\nu g)}}{z(k+\nu g)} du = \int_{2kL+\nu\pi}^{\infty} \frac{e^{i\xi}}{\xi} d\xi$$

As a result

$$\Delta F_\nu = \frac{1}{2}\left(E(2kL+\nu\pi) + (2kL-\nu\pi)\right) \quad E(x) = C\,i(x) + i\,S\,i(x) = -\int_{x}^{\infty} \frac{e^{it}}{t} d\,t \quad (x > 0)$$

$$E(x) = \mathrm{E}^*(\text{-}\,\mathrm{x}) = C\,i(-x) - i\,S\,i(-x) \quad (x < 0)$$

The Fourier theorem gives $V_n = \dfrac{1}{2L} \displaystyle\int_{-L}^{L} V(z) \cos\ n\, gz\; d\, z$ (22)

and after substitution of (14)

$$V_n = \frac{1}{2L} \int\!\!\int_{-L}^{L} f(z-z') \cos n\, g\, z \;\; U(z')\, dz\, dz' \qquad (23)$$

After substitution of the U(z) in the form of Fourier series (20) we obtain the system of linear equations with coefficients D_{nq}:

$$V_n = \sum_q D_{nq} U_q \; ; D_{nq} = \frac{\varepsilon_q}{2L} \int_{-L}^{L}\int f(z-z')\cos(ngz)\ \cos(qgz')\, dz\, dz' \qquad (24)$$

The integral of D_{nq} can be calculated easily by the substitution of series (18) for $f(z-z')$ with subsequent term-by-term integration.

$$D_{nq} = F_n\delta_{nq} + \frac{8}{\pi^2}\varepsilon_q \sum_m \alpha_{nm}\,\alpha_{qm}\,F_m \qquad (25)\quad \text{where}$$

$$\alpha_{nm} = g\int_0^L \cos(ngz)\ \cos(mgz)dz = (-1)^{\frac{n+m+1}{2}} \frac{m}{n^2-m^2} \qquad (26)$$

Here n and q are even values, m and p - odd ones.

Details are quite obvious: we have

$$f = \frac{g}{\pi}\sum_{\nu=0}^{\infty}\varepsilon_\nu F_\nu (\cos(\nu gz)\cos(\nu gz') + \sin(\nu qz)\sin(\nu gz'))$$

and double integral turns into multiplication of two simple ones ,with results proportional to α_{nm}

Obviously, all integrals of sin (ngz) * cos (qgz) and vice versa are equal to zero and integrals of cos (ngz) * cos (qgz) are nonzero if n = q only , and so we come to the formula (25) .

For the even part of V(z) we have

$$V(z) = C^0 V^0(z) + C\ \cos kz$$

The Fourier transform of second is quite simple, and so we have

$$V_n = C^0 V_n^{\,0} + C V_n^{\,'} \qquad V_0' = \frac{\sin kL}{kL} \qquad V_n' = \frac{4}{\pi^2} kL\ \sin kL \frac{(-1)^{\frac{n}{2}}}{\left(\frac{2kL}{\pi}\right)^2 - n^2}$$

On account of (16)

$$V^0(z)=\frac{i}{2}\sin k\left|\ z-z_0\ \right| \qquad V^*{}_n=\frac{1}{2L}\int_{-L}^{L}V^0(z)\ \cos\ ngz\ d\,z \qquad (27)$$

After substitution of first of (27) into the second and integration we get

$$V_n^*\ (z_0)=\left[\frac{4kL\big/\pi^2}{\left(2kL\big/\pi\right)^2-n^2}-\frac{\cos\left((kL+\eta\pi)x_0\right)}{(k+ng)2L}-\frac{\cos\left((kL-\eta\pi)x_0\right)}{(k-ng)2L}\right]\frac{\cos\left(ngz_0\right)}{2}+$$

$$+\frac{\sin\left(ngz_0\right)}{2}\left[\frac{\sin\left(kL+\eta\pi\right)x_0}{(k+ng)2L}-\frac{\sin\left((kL-\eta\pi)x_0\right)}{(k-ng)2L}\right]+$$

$$+\frac{\cos\left(ngz_0\right)}{2}\left[\frac{4kL\big/\pi^2}{\left(2kL\big/\pi\right)^2-n^2}-\frac{\cos\left((kL+\eta\pi)x_1\right)}{(k+ng)2L}-\frac{\cos\left((kL-\eta\pi)x_1\right)}{(k-ng)2L}\right]-$$

$$-\frac{\sin\left(ngz_0\right)}{2}\left[\frac{\sin\left((kL+\eta\pi)x_1\right)}{(k+ng)2L}-\frac{\sin\left(kL-\eta\pi\right)x_1}{(k-ng)2L}\right]$$

And finally, for even part of vector potential

$$V_n^0\ (z_0)\ = 0.5*(\ V_n^*\ (z_0)+V_n^*\ (-z_0)\,)$$

The second solution of (16) , namely , $V^{00}(z)=0.5*\exp(ik\left|\ z-z_0\ \right|)$

differs from the first one by an item of

, $V^{01}(z)=0.5*(\cos(k*(z-z_0))+\cos(k*(z+z_0)))=\cos(kz)*\cos(kz_0)$

Hence the implementation of it differs constant C in vector potential

$V(z)=C^0V^0(z)+C\ \cos kz$

only, and not the current distribution, because k and z_0 are constants .

Here $x_0 = 1 - \frac{z_0}{L}$ $x_1 = 1 + \frac{z_0}{2}$ $\eta = {}^{n}\!/_{2}$

In view of linearity of Maxwell's equations and Fourier transform we may omit factor of i / 2 here and multiply the Green's function by it later The parameter z_0 in the Fourier series equals that in Green's function G (z, z_0)

Indexes of $g^0{}_{ik}$ (6) denote arguments of G (z, z_0) with arbitrary order in view of reciprocity theorem. Boundary conditions give

$$\sum \varepsilon_n (-1)^{n/2} U_n = 0 \text{ or } U_0 = -2\sum_n{}' (-1)^{n/2} U_n \qquad (28)$$

Here and later " shaded " sum means no zero term .For 2L periodic U(z) expansion in " half -integer " Fourier-like series turn into expansion into odd harmonics

$$U(z) = \sum_m \varepsilon_m U_m \cos\ mgz \quad \text{m=1,3,5....} \qquad (29)$$

quite analogous to the way the ordinary Fourier series is replaced by an expansion in even harmonics n or q $U(z) = \sum_n \varepsilon_n U_n \cos(n gz)$ (20)

and coefficients U_m ,U_n obey a simple relationships

Analogous to $U_n = \frac{1}{2L}\int_{-L}^{L} U(z) \cos\ n gz\, d z$ (30)

for odd harmonics we have $U_m = \frac{1}{2L}\int_{-L}^{L} U(z) \cos\ m gz\, d z$. (31)

Substituting $U(z) = \sum_n \varepsilon_n U_n \cos(n gz)$ into (31) we get

$$U_m = \frac{2}{\pi}\sum_n \alpha_{nm}\, \varepsilon_n U_n , \qquad (32)$$

and substituting (29) into (30) we find $U_n = \frac{2}{\pi}\sum \alpha_{nm}\,\varepsilon_m\,U_m$ (33)

These relations along with some symmetrical ones to follow we need to establish the plentitude of Fourier- like series in odd harmonics.

For odd parts of vector potential V(z) and current U(z) we write expansions in odd harmonics m and p

$$V(z) = \sum_m \varepsilon_m V_m \sin\, mgz \qquad U(z) = \sum_p \varepsilon_p U_p \sin(pgz) \qquad (34)$$

and, following the logic outlined in the calculations for even part we have a system of linear equations $V_m = \sum \mathrm{E}_{mp}\,U_p$ (35) where

$$\mathrm{E}_{mp} = F_m \delta_{mp} + \frac{8}{\pi^2}\sum_n \beta_{nm}\,\beta_{np}\,\varepsilon_n\,F_n \qquad (36)$$

$$\beta_{nm} = g\int_0^L \sin ngz\ \sin mgz\ dz = (-1)^{\frac{n+m+1}{2}}\,\frac{n}{n^2 - m^2} \qquad (37)$$

with vector potential $V(z) = C^0\,V^0(z) + C\sin kz$

and Fourier coefficients

$$V_m = C^0\,V_m{}^0 + C\,V'_m \qquad V'_m = \frac{4}{\pi^2}\,k\,L\ \cos kL\ \frac{(-1)^{\frac{m+1}{2}}}{\left(\frac{2kL}{\pi}\right)^2 - m^2}$$

$$V_m^{*}(z_0) = \left[\frac{4kL/\pi^2}{\left(2kL/\pi\right)^2 - m^2} - \frac{\cos((kL+\mu\pi)x_0)}{(k+mg)2L} - \frac{\cos((kL-\mu\pi)x_0)}{(k-mg)2L} \right] \frac{\sin(mgz_0)}{2} -$$

$$-\frac{\cos(mgz_0)}{2}\left[\frac{\sin((kL+\mu\pi)x_0)}{(k+mg)2L} - \frac{\sin((kL-\mu\pi)x_0)}{(k-mg)2L} \right] +$$

$$+\frac{\sin(mgz_0)}{2}\left[\frac{4kL/\pi^2}{\left(2kL/\pi\right)^2 - m^2} - \frac{\cos((kL+\mu\pi)x_1)}{(k+mg)2L} - \frac{\cos((kL-\mu\pi)x_1)}{(k-mg)2L} \right] +$$

$$+\frac{\cos(mgz_0)}{2}\left[\frac{\sin((kL+\mu\pi)x_1)}{(k+mg)2L} - \frac{\sin((kL-\mu\pi)x_1)}{(k-mg)2L} \right]$$

and , as a result, $V_m^{0}\ (z_0)\ = 0.5*(\ V_m^{*}\ (z_0) - V_m^{*}\ (-z_0))$

The second solution of (16) , namely , $V^{00}(z) = 0.5 * \exp(ik|\ z - z_0\ |)$

differs from the first one by an item of

$$V^{01}(z) = 0.5 * (\cos(k*(z-z_0)) - \cos(k*(z+z_0))) = \sin(kz) * \sin(kz_0)$$

Hence the implementation of it differs the constant C in vector

potential $V(z) = C^0 V^0(z) + C\ \sin(kz)$

only, and not the current distribution, because k and z_0 are constants .

here $x_0 = 1 - \frac{z_0}{L}$ $\qquad x_1 = 1 + \frac{z_0}{2}$ $\qquad \mu = \frac{m}{2}$ = (m-1) / 2 + 0.5.

Boundary condition (zero current at the ends) give $\sum (-1)^{\frac{m+1}{2}}\ U_m = 0$

Ordinary Fourier expansion of odd function U(z) is as follows

$U(z) = \sum_n \varepsilon_n U_n\ \sin(ngz)$ Here $U_n = \frac{1}{2L}\int_{-L}^{L} U(z)\ \sin\ ngz\ d\,z$ (38)

and comparing this to the Fourier-like expansion in odd harmonics (second formula of 34 with m instead of p) we find

$$U_m = \frac{2}{\pi}\sum_n \varepsilon_n \beta_{nm} U_n \qquad U_n = \frac{2}{\pi}\sum \beta_{nm} \varepsilon_m U_m \quad (39)$$

So ordinary Fourier series with the even harmonics due to choice

$g = \dfrac{\pi}{2L}$ and Fourier like expansion in odd harmonics are

equivalent., and by comparing them we get a sort of orthogonality relations

$$\frac{4}{\pi^2}\varepsilon_m \sum_n \varepsilon_n \alpha_{nm} \alpha_{np} = \frac{4}{\pi^2}\varepsilon_m \sum_n \varepsilon_n \beta_{nm} \beta_{np} = \delta_{mp} \qquad (40)$$

$$\frac{4}{\pi^2}\varepsilon_n \sum_m \varepsilon_m \alpha_{nm} \alpha_{qm} = \frac{4}{\pi^2}\varepsilon_n \sum_m \varepsilon_m \beta_{nm} \beta_{qm} = \delta_{nq} \qquad (41)$$

Putting (33) into (32) with replacement p->m and interchanging summations we get the first of (40)

Similarly, putting second of (39) into the first we get the second of (40)

First of (41) we get after putting (32) into(33) with replacement q->n and second of (39) put into the first gives the second of (41) the same way

The characteristic of Uda- Yagi antenna is the so called " shield coefficient " i.e. the ratio of forward to backvard wave amplitudes.

Our computations showed that for the accuracy of shield coeficient within 1 decibel the Fourier series of about 3000 harmonics should be put into calculations.

Hence the implementation of the LU decomposition with the Crout method look dubious. . It occurs that iteration procedure leads much faster to solution

A little transformation of (24) indicates the first iterations to employ

For odd and even parts we have.

$$D_{nq} = 2F_n\delta_{nq} + d_{nq}$$

$$d_{nq} = \frac{8}{\pi^2}\varepsilon_q \sum_m \alpha_{nm}\,\alpha_{qm}\,F_m - F_n\,\delta_{nq} = \frac{8}{\pi^2}\;\varepsilon_q \sum_m \alpha_{nm}\,\alpha_{qm}\left(F_m - F_n\right)$$

$$\mathrm{E}_{mp} = 2F_m\delta_{mp} + e_{mp}$$

$$e_{mp} = \frac{8}{\pi^2}\sum_n \varepsilon_n\,\beta_{nm}\;\beta_{np}\,F_n - F_m\,\delta_{mp} = \frac{16}{\pi^2}\;\sum_n \beta_{nm}\,\beta_{np}\left(F_n - F_m\right)$$

Orthogonality relations , used in the last equations of both, valid for infinite Fourier and odd Fourier-like series, are used to estimate the length of series required for the desired precision to attain.

It appears that for the 8-digits precision we need finite Fourier series length of 5000 for F_n and F_m both, and, as final results showed, about 3000 -length Fourier series for " n " and " m " in

$$V_n = 2F_n\,U_n + \sum_q d_{nq}\,U_q \qquad V_m = 2F_m\,U_m + \sum_p e_{mp}\,U_p$$

to attain the accuracy within 1 decibel for shield coefficient above.

First iteration to take is that as if sums above are zero both, and we iterate these sums in both equations (series U_n and U_m to be defined)

Constants C in expressions for vector potential provide the zero of both even and odd currents on the ends but now they prevent us from implementing the technique outlined directly . So we use the superposition principle and write current distribution as follows,

for even part first : $U_n = C^0\,U_n^0 + CU'_n$

and we get two linear systems with no undefined constants

$$U_n^0 = \frac{1}{2F_n}\left(V_n^0 - \sum d_{nq}\,U_q^0\right) \qquad U_n^{/} = \frac{1}{2F_n}\left(V_n^{/} - \sum d_{nq}\,U_q^{'}\right)$$

First one represents current wave induced by EMF and second - reflected one

Boundary conditions give us then

$$C = -C^0\,\frac{\sum\limits_n \varepsilon_n (-1)^{n/2} U_n^0}{\sum\limits_n \varepsilon_n (-1)^{n/2} U_n'}$$

where C^0 may be set equal to 1 with no loss of generality.

For odd part of solution we have $U_m = C^0\,U_m^0 + C\,U_m'$

$$U_m^0 = \frac{1}{2F_m}\left(V_m^0 - \sum_p e_{mp}\,U_p^0\right) \qquad U'_m = \frac{1}{2F_m}\left(V_m' - \sum e_{mp}\,U_p^{'}\right)$$

$$C = -C^0\,\frac{\sum\limits_m (-1)^{\frac{m+1}{2}}\,U_m^0}{\sum (-1)^{\frac{m+1}{2}}\,U^{/}{}_m}$$

Solution procedure outlined converges to computer accuracy in 10-15 iterations . Some other transformations of (24) are also possible and proven equivalent, for instance $U_n = C^0\,U_n^0 + CU_n' + U_0\,U_n{}^{'''}$ with zero even harmonic as a new variable. and " n " non-zero. Specifically,

for even part we have

$$V_0 = C^0 V^0{}_0 + CV^{'}{}_0 = D_{00}\ U_0 + \sum_n{}' D_{0q}\,U_q \ ;\ V_n = 2F_n\ U_n + d_{n0}\,U_0 + \sum_q{}' d_{nq}\,U_q$$

here $U_q = C^0\,U_q^0 + CU_q' + U_0\,U_q{}^{'''}$. Hence we obtain

$$V_n = C^0 V_n{}^0 + CV_n{}^{'} = 2F_n(C^0U^0{}_n + CU_n' + U_0U_n{}^{''}) + d_{n0}U_0 + \sum_q{}' C^0 d_{nq}\,U^0{}_q\ +$$

$\sum\limits_q{}' C\ d_{nq}\,U^{'}{}_q + \sum\limits_q{}' U_0\ d_{nq}\,U^{''}{}_q$ And come to equations for new variables

$$U_n^0 = \frac{1}{2F_n}\left(V_n^0 - \sum_q{}' d_{nq}U_q^0\right)$$

$$U_n^{'} = \frac{1}{2F_n}\left(V_n^{'} - \sum_q{}' d_{nq}U_q^{'}\right)$$

$$U_n^{''} = \frac{1}{2F_n}\left(-d_{n0} - \sum_q{}' d_{nq}U_q^{''}\right)$$

Zero harmonics, not used in these equations, are related as above:

$$C^0\sum\nolimits_n' D_{0n}U_n^0 + C\left[\sum_n{}' D_{0n}U_n^{'}\right] + U_0\left[\sum_n{}' D_{0n}U_n^{''} + D_{00}\right] = C^0\left[V_0^0\right] + CV'_0 \qquad (42)$$

And boundary condition takes now the form

$$C\sum\nolimits_n{}'(-1)^{n/2}U_n^{'} + U_0\left[\sum\nolimits_n{}'(-1)^{n/2}U_n^{''} + \frac{1}{2}\right] = -C^0\sum\nolimits_n'(-1)^{n/2}U_n^0 \qquad (43)$$

and these equations, after getting solutions of three linear system above, give us two constants C and U_0 not defined yet.

For some more ease in calculations we rewrite (42) as

$$C\left[\sum_n{}' D_{0n}U_n^{'} - V_0^{'}\right] + U_0\left[\sum_n{}' D_{0n}U_n^{''} + D_{00}\right] = -C^0\left[\sum\nolimits_n' D_{0n}U_n^0 - V_0^0\right] \qquad (44)$$

This alternative method obviously applies to the even part of current distribution only and gives, within computer accuracy of 15 digits, the very same results for both current distribution and vector potential (the solution of . of (43) and (44) is found immediately) .

The calculation is completed by calculating the current distribution in the form of Fourier series for even part and Fourier - like series for odd, and multiplying the sum by a factor of 0.5 * i as noted above.

External current I_0^e in the emitter (12), created by EMF equal 1 is the

Green's function with $z_0 = 0$, and now we get a solution of (7)- (12) .

This completes the task of finding current distributions and we go to field proceedings. The Green's functions found obey the reciprocity theorem with computer accuracy, currents calculated from (7)- (12) are zeros at the ends, and, with center fed emitter , current distributions appear to be symmetrical if antenna elements (emitter, reflector and directors) are center- aligned.

The computer program based on this method completed all the calculations in less than an hour with 2.8 GHZ four-head processor. Six--head processor is expected to almost halve this time consumption as a little consideration shows immediately .

For some practical Uda-Yagi antenna with elements chosen by MMANA program which gave shield coefficient of 20 decibels we succeeded in establishing shield coefficient of above 40 decibels, by small corrections of the antenna elements lengths only.

Appendix. .

1 Let's calculate constant A in (13-0). For the sake of brevity assume $k = r = 0$. Infeld function equals 1 then and for $z = z'$

$f = \frac{1}{a} = A_0 \int_0^{\infty} K_0(wa)dw$ We have an integral representation :

$K_0(x) = \int_0^{\infty} \exp(-x * ch(t))dt$ Substituting into $f = \frac{1}{a} = \frac{A_0}{a} \int_0^{\infty} K_0(x)dx$

and integrating

$$\frac{1}{A_0} = \int_0^\infty\int_0^\infty \exp(-x * ch(t))\, dx\, dt = \int_0^\infty \frac{dt}{ch(t)} = 2\int_1^\infty \frac{dy}{(y^2+1)} \quad (y = \exp(t))$$

Last one equals $\frac{\pi}{4}$ and $A_0 = \frac{2}{\pi}$, accordingly. For symmetrical integration limits $A = \frac{1}{\pi}$. So (13) appears to be a reverse Fourier transform of even function $f(|z - z'|)$

2 Let's give a Gauss elimination details of (7)-(12).

It's common that equalities like A = A + B * C (A, B, C- square matrices) mean calculation of right-hand side and substitution the result into matrix A.

E - identity matrix, i.e. Kronecker delta- symbol.

First, we eliminate I_i^l in (8) substituting (7) instead :

$$tr_{ij} = tr_{ij} + tl_{ik} \times lr_{kj} \quad tf_{ij} = tf_{ij} + tl_{ik} + lf_{kj} \quad te_{ij} = te_{ij} + tl_{ik} \times le_{kj}$$

$$ts_{ij} = ts_{ij} + tl_{ik} \times ls_{kj} \quad tt_{ij} = tl_{ik} \times lt_{kj}$$

Now, transfer tt_{ij} to the LHS and calculating the reverse of matrix there,

$a\theta_{ij} = \left(E_{ij} - tt_{ij}\right)^{-1}$, after multiplying matrices in the RHS by it

$$tr_{ij} = a\theta_{ik} \times tr_{kj} \quad te_{ij} = a\theta_{ik} \times te_{kj} \quad tf_{ij} = a\theta_{ik} \times tf_{kj} \quad ts_{ij} = a\theta_{ik} \times ts_{kj}$$

we have instead of (8)

$$I_i^t = (ts)_{ij} I_j^s + (tf)_{ij} I_j^f + (tr)_{ij} I_j^r + (te)_{ij} I_j^e \qquad (8\text{-a})$$

Now, in the same way we eliminate I_i^l in (9) substituting (7) instead :

$$sr_{ij} = sr_{ij} + sl_{ik} \times lr_{kj} \quad se_{ij} = se_{ij} + sl_{ik} \times le_{kj} \quad sf_{ij} = sf_{ij} + sl_{ik} \times lf_{kj}$$

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$st_{ij} = st_{ij} + sl_{ik} \times lt_{kj}$ $ss_{ij} = sl_{ik} \times ls_{kj}$

and I_i^t, substituting (8-a) :

$sr_{ij} = sr_{ij} + st_{ik} \times tr_{kj}$ $se_{ij} = se_{ij} + st_{ik} \times te_{kj}$

$sf_{ij} = sf_{ij} + st_{ik} \times tf_{kj}$ $ss_{ij} = ss_{ij} + st_{ik} \times ts_{kj}$

then transfer ss_{ij} to the LHS and after reversing the matrix there

$a\theta_{ij} = \left(E_{ij} - ss_{ij}\right)^{-1}$ after multiplying matrices in RHS by it

$sr_{ij} = a\theta_{ik} \times sr_{kj}$ $se_{ij} = a\theta_{ik} \times se_{kj}$ $sf_{ij} = a\theta_{ik} \times sf_{kj}$

we have instead of (9)

$I_i^s = (sf)_{ij} I_j^f + (sr)_{ij} I_j^r + (se)_{ij} I_j^e$ (9-a)

As before, substituting into (10) equations (7) :

$fr_{ij} = fr_{ij} + fl_{ik} \times lr_{kj}$ $fe_{ij} = fe_{ij} + fl_{ik} \times le_{kj}$ $ft_{ij} = ft_{ij} + fl_{ik} \times lt_{kj}$

$fs_{ij} = fs_{ij} + fl_{ik} \times ls_{kj}$ $ff_{ij} = fl_{ik} \times lf_{kj}$,

(8-a) :

$fr_{ij} = fr_{ij} + ft_{ik} \times tr_{kj}$ $fe_{ij} = fe_{ij} + ft_{ik} \times te_{kj}$

$fs_{ij} = fs_{ij} + ft_{ik} \times ts_{kj}$ $ff_{ij} = ff_{ij} + ft_{ik} \times tf_{kj}$

and (9-a) :

$fr_{ij} = fr_{ij} + fs_{ik} \times sr_{kj}$ $fe_{ij} = fe_{ij} + fs_{ik} \times se_{kj}$ $ff_{ij} = ff_{ij} + fs_{ik} \times sf_{kj}$

and, after that, transferring ff_{ij} to the RHS of (10) rewrite it as follows

$a\theta_{ij} = \left(E_{ij} - ff_{ij}\right)^{-1}$ $fr_{ij} = a\theta_{ik} \times fr_{kj}$ $fe_{ij} = a\theta_{ik} \times fe_{kj}$ we get

$I_i^f = (fr)_{ij} I_j^r + (fe)_{ij} I_i^e$ (10-a)

Next, substituting into (11) equations (7) :

$$re_{ij} = re_{ij} + rl_{ik} \times le_{kj} \quad rf_{ij} = rf_{ij} + rl_{ik} \times lf_{kj} \quad rs_{ij} = rs_{ij} + rl_{ik} \times ls_{kj}$$

$$rt_{ij} = rt_{ij} + rl_{ik} \times lt_{kj} \quad rr_{ij} = rl_{ik} \times lr_{kj}$$

(8-a) :

$$rs_{ij} = rs_{ij} + rt_{ik} \times ts_{kj} \quad rf_{ij} = rf_{ij} + rt_{ik} \times tf_{kj}$$

$$re_{ij} = re_{ij} + rt_{ik} \times te_{kj} \quad rr_{ij} = rr_{ij} + rt_{ik} \times tr_{kj}$$

(9-a) :

$$rf_{ij} = rf_{ij} + rs_{ik} \times sf_{kj} \quad re_{ij} = re_{ij} + rs_{ik} \times se_{kj} \quad rr_{ij} = rr_{ij} + rs_{ik} \times sr_{kj}$$

and (10-a) :

$$re_{ij} = re_{ij} + rf_{ik} \times fe_{kj} \qquad rr_{ij} = rr_{ij} + rf_{ik} \times fr_{kj}$$

rewrite (11) by LHS matrix reversion

$a\theta_{ij} = \left(E_{ij} - rr_{ij}\right)^{-1} \quad re_{ij} = a\theta_{ik} \times re_{kj}$ in the form

$I_i^r = \left(re\right)_{ij} I_j^e$ (11-a)

and, finally, substituting into (12) equations (7) :

$$er_{ij} = er_{ij} + el_{ik} \times lr_{kj} \quad ef_{ij} = ef_{ij} + el_{ik} \times lf_{kj} \quad es_{ij} = es_{ij} + el_{ik} \times ls_{kj}$$

$$et_{ij} = et_{ij} + el_{ik} \times lt_{kj} \quad ee_{ij} = el_{ik} \times le_{kj}$$

(8-a) :

$$es_{ij} = es_{ij} + et_{ik} \times ts_{kj} \quad ef_{ij} = ef_{ij} + et_{ik} \times tf_{kj}$$

$$er_{ij} = er_{ij} + et_{ik} \times tr_{kj} \quad ee_{ij} = ee_{ij} + et_{ik} \times te_{kj}$$

(9-a) :

$$ef_{ij} = ef_{ij} + es_{ik} \times sf_{kj} \quad er_{ij} = er_{ij} + es_{ik} \times sr_{kj} \quad ee_{ij} = ee_{ij} + es_{ik} \times se_{kj}$$

(10-a) :

$$er_{ij} = er_{ij} + ef_{ik} \times fr_{kj} \quad ee_{ij} = ee_{ij} + ef_{ik} \times fe_{kj}$$

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11-a :

$$ee_{ij} = ee_{ij} + er_{ik} \times re_{kj}$$

we obtain for the center-fed emitter

$(E - ee)_{ij} I_j^e = I_i^0$ here E - unity matrix.

Right-hand side represents current created by external EMF . Hence this equation for the total current in the emitter is solved immediately

By means of back substitutions in equations (11-a) (10-a) (9-a) (8-a) and (7) we find all currents as follows.

$$I_i^r = re_{ij} \times I_j^e$$

$$I_i^f = fe_{ij} \times I_j^e + fr_{ij} \times I_j^r \qquad I_i^s = se_{ij} \times I_j^e + sr_{ij} \times I_j^r + sf_{ij} \times I_j^f$$

$$I_i^t = te_{ij} \times I_j^e + tr_{ij} \times I_j^r + tf_{ij} \times I_j^f + ts_{ij} \times I_j^s$$

$$I_i^l = le_{ij} \times I_j^e + lr_{ij} \times I_j^r + lf_{ij} \times I_j^f + ls_{ij} \times I_j^s + lt_{ij} \times I_j^t$$

Substitution this into original equations (7) - (12) showed that

the solution obeys them with computer accuracy , appears symmetrical in case of center-fed emitter and elements center-aligned, as it should've been.

3

Let us calculate the Fourier and Fourier- like transforms of vector potential (16)

Rewriting the first one in the form we obtain.

$$V^*_n = \frac{1}{2L}\int_{z_0}^{L} \sin k(z - z_0)\ \cos\ ngz\ dz - \frac{1}{2L}\int_{-L}^{z_0} \sin k(z - z_0)\ \cos\ ngz\ dz; z - z_0 = \pm x$$

$$V^*{}_n = \frac{1}{2L}\int_0^{L-z_0} \sin kx\ \cos\ n\,g(x+z_0)\,d\,x + \frac{1}{2L}\int_0^{L+z_0} \sin k(x)\ \cos\ n\,g(x-z_0)\,dx;$$

On account of sin(kx) cos(ngx)=0.5*(sin(k+ng)x+sin(k-ng)x)

and -sin(kx) sin(ngx)=0.5*(cos(k+ng)x-cos(k-ng)x)

we have

$$V_n^*\ (z_0) = \left[\frac{{}^{4kL}\!/_{\pi^2}}{\left({}^{2kL}\!/_{\pi}\right)^2 - n^2} - \frac{\cos\left((kL+\eta\,\pi)x_0\right)}{(k+ng)2L} - \frac{\cos\left((kL-\eta\pi)x_0\right)}{(k-ng)2L}\right]\frac{\cos\left(ngz_0\right)}{2} +$$

$$+\frac{\sin\left(ngz_0\right)}{2}\left[\frac{\sin\left(kL+\eta\pi\right)x_0}{(k+ng)2L} - \frac{\sin\left((kL-\eta\,\pi)x_0\right)}{(k-ng)2L}\right] +$$

$$+\frac{\cos\left(ngz_0\right)}{2}\left[\frac{{}^{4\,kL}\!/_{\pi^2}}{\left({}^{2\,kL}\!/_{\pi}\right)^2 - n^2} - \frac{\cos\left((kL+\eta\,\pi)x_1\right)}{(k+ng)2L} - \frac{\cos\left((kL-\eta\pi)x_1\right)}{(k-ng)2L}\right] -$$

$$-\frac{\sin\left(n\,g\,z_0\right)}{2}\left[\frac{\sin\left((kL+\eta\pi)x_1\right)}{(k+n\,g)2L} - \frac{\sin\left(kL-\eta\pi\right)x_1}{(k-ng)2L}\right]$$

Fourier -like expansion in odd harmonics appears to be

$$V^*{}_m(z_0) = \frac{1}{2L}\int_{z_0}^{L} \sin k(z-z_0)\ \sin\ \ m\,gz\ d\,z + \frac{1}{2L}\int_{-L}^{z_0} \sin k(z_0-z)\ \sin\ \ m\,gz\ d\,z; z-z_0 = \pm x$$

$$V^*{}_m(z_0) = \frac{1}{2L}\int_0^{L-z_0} \sin kx\ \sin\ \ m\,g(x+z_0)\,d\,x - \frac{1}{2L}\int_0^{L+z_0} \sin k(x)\ \sin\ \ m\,g(x-z_0)\,dx;$$

and on account of sin(kx)cos(mgx)=0.5*(sin(k+mg)x+sin(k-mg)x)

and -sin(kx)sin(mgx)=0.5*(cos(k+mg)x-cos(k-mg)x)

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we find

$$V_m^{*}(z_0)=\left[\frac{4kL/\pi^2}{\left(2kL/\pi\right)^2-m^2}-\frac{\cos\left((kL+\mu\pi)x_0\right)}{(k+mg)2L}-\frac{\cos\left((kL-\mu\pi)x_0\right)}{(k-mg)2L}\right]\frac{\sin\left(mgz_0\right)}{2}-$$

$$-\frac{\cos\left(mgz_0\right)}{2}\left[\frac{\sin\left((kL+\mu\pi)x_0\right)}{(k+mg)2L}-\frac{\sin\left((kL-\mu\pi)x_0\right)}{(k-mg)2L}\right]+$$

$$+\frac{\sin\left(mgz_0\right)}{2}\left[\frac{4kL/\pi^2}{\left(2kL/\pi\right)^2-m^2}-\frac{\cos\left((kL+\mu\pi)x_1\right)}{(k+mg)2L}-\frac{\cos\left((kL-\mu\pi)x_1\right)}{(k-mg)2L}\right]+$$

$$+\frac{\cos\left(mgz_0\right)}{2}\left[\frac{\sin\left((kL+\mu\pi)x_1\right)}{(k+mg)2L}-\frac{\sin\left((kL-\mu\pi)x_1\right)}{(k-mg)2L}\right]$$

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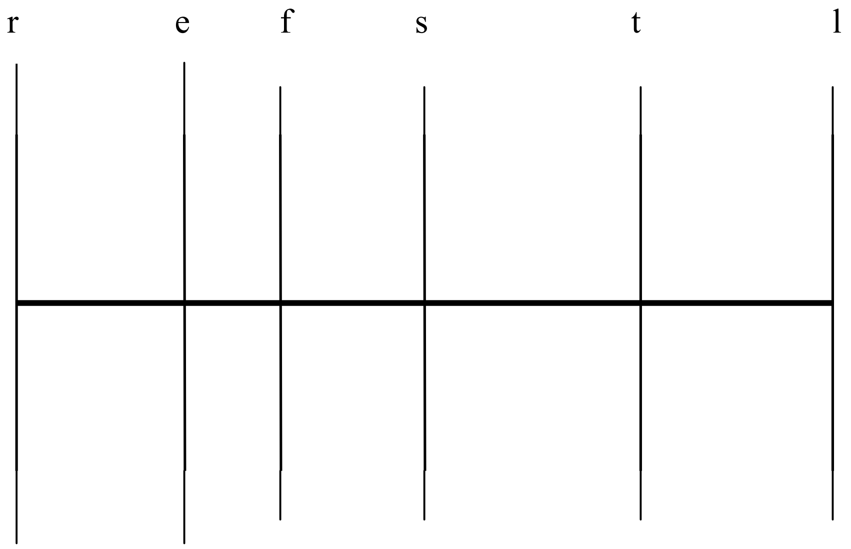
4) Ch Golub L Van Loan Matrix computations.



Uda-Yagi antenna image . r- reflector, e- center- fed emitter f , s, t, l - directors, from first to fourth (last).